Traveling wave solutions of the BBM-like equations

2009 J. Phys. A: Math. Theor. 42 375203

(http://iopscience.iop.org/1751-8121/42/37/375203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 375203 (12pp) doi:10.1088/1751-8113/42/37/375203

Traveling wave solutions of the BBM-like equations

Ş Kuru

Department of Physics, Faculty of Science, Ankara University 06100 Ankara, Turkey

E-mail: kuru@science.ankara.edu.tr

Received 26 May 2009, in final form 3 August 2009
Published 25 August 2009
Online at stacks.iop.org/JPhysA/42/375203

Abstract
In this work, we apply the factorization technique to the Benjamin–Bona–Mahony-like equations, $B(m,n)$, in order to get traveling wave solutions. We will focus on some special cases for which $m \neq n$, and we will obtain these solutions in terms of the special forms of Weierstrass functions.

PACS numbers: 05.45.Yv, 02.30.Jr

1. Introduction

In this paper, we will consider the Benjamin–Bona–Mahony (BBM) [1] like equations with a fully nonlinear dispersive term of the form

$$u_t + u_x + a(u^m)_x - (u^n)_{xxt} = 0, \qquad m, n > 1, \quad m \neq n. \tag{1.1}$$

This equation is similar to the nonlinear dispersive equation $K(m,n)$,

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \qquad m > 0, \quad 1 < n \leqslant 3, \tag{1.2}$$

which has been studied in detail by Rosenau and Hyman [2]. These kinds of nonlinear equations have recently attracted much attention due to the existence of solitary wave solutions with compact support called compactons. In the literature, there are many studies dealing with the traveling wave solutions of the $K(m,n)$ and $B(m,n)$ equations, but in general they are restricted to the case $m = n$ [2–16]. Our aim here is just to search for the traveling wave solutions of the $B(m,n)$ equations, with $m \neq n$, by means of the factorization technique [17–21]. We remark that this method allows us to get a wider set of solutions, compared with other methods, such as the sine–cosine and the tanh methods, the extended Riccati equations method, Jacobian elliptic function expansion method [11, 12, 22–24], used to solve BBM equations. In [14], the symmetry reductions of $B(m,n)$ equations have been derived and also some exact solutions have been obtained for special values of the parameters. It is clear that one of the symmetry reductions of $B(m,n)$ supplied in [14] coincides with the traveling wave reduction of the $B(m,n)$ equations. However, here we will get a wider class of solutions, for this reduction, than those presented in [14]. We note that the direct integral method [26] can also be used for $B(m, n)$ equations, with $m \neq n$. In general, the factorization technique enables us to solve a large set of nonlinear equations in comparison with this method. We remark that this technique has been used to obtain solutions of the second-order ordinary differential equations (ODEs) with variable coefficients [20] and it has been recently extended to third-order nonlinear ODEs by Wang and Li [25].

When we look for the traveling wave solutions of equation (1.1), first we reduce the form of the $B(m, n)$ equation to a second-order nonlinear ODE and then this equation can be factorized in two ways. The first one is to use the factorization technique to factorize it in terms of differential operators. The second one is to use the usual way (see section 2) and factorize a constant of motion (identified with energy) as the product of two equations. These two factorizations give rise to the same first-order ODE that allow us to get the traveling wave solutions of the $B(m, n)$ equation. Here, we will assume $m \neq n$, since the case $m = n$ has already been examined in a previous article following this method [16].

This paper is organized as follows. In section 2, we introduce the factorization technique for a special type of second-order nonlinear ODEs. Then, we apply it straightforwardly to the reduced second-order ODE of the $B(m, n)$ equation in order to get the traveling wave solutions. We obtain the solutions for this nonlinear ODE and the $B(m, n)$ equation in terms of Weierstrass functions in section 3. In section 4 we will find the constant of motion interpreted as the energy and how its factorization gives the factorized equations of section 2. Finally, section 5 will end this work with some conclusions and remarks.

2. Factorization of the BBM-like equations

2.1. Factorization of nonlinear second-order ODEs

Let us consider the following nonlinear second-order ODE:

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - \beta \frac{\mathrm{d}W}{\mathrm{d}\theta} + F(W) = 0, \tag{2.1}$$

where β is constant and $F(W)$ is an arbitrary function of W. The factorized form of this equation can be written as

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta} - f_2(W, \theta)\right]\left[\frac{\mathrm{d}}{\mathrm{d}\theta} - f_1(W, \theta)\right] W(\theta) = 0. \tag{2.2}$$

Here, f_1 and f_2 are unknown functions that may depend explicitly on W and θ. Expanding (2.2) and comparing with (2.1), we obtain the following consistency conditions:

$$f_1 f_2 = \frac{F(W)}{W} + \frac{\partial f_1}{\partial \theta}, \qquad f_2 + \frac{\partial (W f_1)}{\partial W} = \beta. \tag{2.3}$$

To solve (2.3) for f_1 or f_2 we will supply a compatible first-order ODE,

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta} - f_1(W, \theta)\right] W(\theta) = 0, \tag{2.4}$$

that provides a solution to the nonlinear ODE (2.1) [17–20]. In the applications of this paper f_1 and f_2 will depend only on W.

2.2. Factorization of the B(m,n) equations

If equation (1.1) has the traveling wave solutions in the form

$$u(x, t) = \phi(\xi), \qquad \xi = hx + wt, \tag{2.5}$$

where h and w are the real constants, substituting (2.5) into (1.1) and after integrating, we get the traveling wave reduction

$$(\phi^n)_{\xi\xi} - A\phi - B\phi^m + D = 0, \tag{2.6}$$

which is a second-order nonlinear ODE. Note that the constants in equation (2.6) are

$$A = \frac{h+w}{h^2 w}, \qquad B = \frac{a}{hw}, \qquad D = \frac{R}{h^2 w} \tag{2.7}$$

and R is an integration constant. Now, we introduce the following natural transformation of the dependent variable:

$$\phi^n(\xi) = W(\theta), \qquad \xi = \theta, \tag{2.8}$$

so that equation (2.6) becomes

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - AW^{\frac{1}{n}} - BW^{\frac{m}{n}} + D = 0. \tag{2.9}$$

At this moment, we can apply easily the factorization technique to this equation. Comparing equation (2.1) and equation (2.9), we have $\beta = 0$ and

$$F(W) = -(AW^{\frac{1}{n}} + BW^{\frac{m}{n}} - D). \tag{2.10}$$

Replacing these expressions in (2.3) we get only one consistency condition

$$f_1^2 + f_1 W \frac{\mathrm{d}f_1}{\mathrm{d}W} - AW^{\frac{1-n}{n}} - BW^{\frac{m-n}{n}} + DW^{-1} = 0 \tag{2.11}$$

whose solutions are

$$f_1(W) = \pm \frac{1}{W} \sqrt{\frac{2nA}{n+1} W^{\frac{n+1}{n}} + \frac{2nB}{m+n} W^{\frac{m+n}{n}} - 2DW + C}, \tag{2.12}$$

where C is an integration constant. Thus, the first-order ODE (2.4) takes the form

$$\frac{\mathrm{d}W}{\mathrm{d}\theta} \mp \sqrt{\frac{2nA}{n+1} W^{\frac{n+1}{n}} + \frac{2nB}{m+n} W^{\frac{m+n}{n}} - 2DW + C} = 0. \tag{2.13}$$

This result, (2.13), can also be obtained in the usual way for ODEs by multiplying the autonomous equation (2.9) by $\left(2\frac{\mathrm{d}W}{\mathrm{d}\theta}\right)$ and then integrating once since in this case $\beta = 0$. In fact, this way will be illustrated in section 4, where it is discussed the constant of motion identified with energy. Thus, we can factorize our second-order equation in terms of two equations given in (2.13) with $C = 2E$ where E is the constant of motion.

In order to solve equation (2.13) for W in a more general way, let us take W in the form $W = \varphi^p$, $p \neq 0$. Then, the first-order ODE (2.13) is rewritten in terms of φ as

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{2nA}{p^2(n+1)} \varphi^{p(\frac{1-n}{n})+2} + \frac{2nB}{p^2(m+n)} \varphi^{p(\frac{m-n}{n})+2} - \frac{2D}{p^2}\varphi^{2-p} + \frac{C}{p^2}\varphi^{2-2p}. \tag{2.14}$$

If we want to guarantee the integrability of (2.14), the powers of φ have to be integer numbers between 0 and 4 [27]. Having in mind the conditions on n, m ($n \neq m$, with $m, n > 1$) and p ($p \neq 0$), we have the following possible cases.

- *Case 1*. If $C = D = 0$, we can choose p and m in the following way:

$$p = \pm \frac{2n}{1-n} \qquad \text{with} \quad m = \frac{n+1}{2}, \frac{3n-1}{2}, 2n-1 \tag{2.15}$$

and

$$p = \pm \frac{n}{1-n} \qquad \text{with} \quad m = 2n-1, 3n-2. \tag{2.16}$$

It can be checked that both signs in (2.15) and (2.16) lead to the same solutions for equation (1.1); henceforth, we will consider only one of them. Thus, for $p = -\frac{2n}{1-n}$, equation (2.14) becomes

$$(1.a)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{n(3n+1)}\varphi, \qquad m = \frac{n+1}{2} \tag{2.17}$$

$$(1.b)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{n(5n-1)}\varphi^3, \qquad m = \frac{3n-1}{2} \tag{2.18}$$

$$(1.c)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{n(3n-1)}\varphi^4, \qquad m = 2n-1 \tag{2.19}$$

and for $p = -\frac{n}{1-n}$, it becomes

$$(1.d)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{2A(n-1)^2}{n(n+1)}\varphi + \frac{2B(n-1)^2}{n(3n-1)}\varphi^3, \qquad m = 2n-1 \tag{2.20}$$

$$(1.e)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{2A(n-1)^2}{n(n+1)}\varphi + \frac{B(n-1)^2}{n(2n-1)}\varphi^4, \qquad m = 3n-2. \tag{2.21}$$

- *Case 2.* If $C = 0$, we have the special cases, $p = \pm 2$, $n = 2$ with $m = 3, 4$.
  Due to the same reason as in the above case, we will consider only $p = 2$. Then, equation (2.14) takes the form

$$(2.a)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -\frac{D}{2} + \frac{A}{3}\varphi + \frac{B}{5}\varphi^3, \qquad m = 3 \tag{2.22}$$

$$(2.b)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -\frac{D}{2} + \frac{A}{3}\varphi + \frac{B}{6}\varphi^4, \qquad m = 4. \tag{2.23}$$

- *Case 3.* If $A = C = 0$, we have $p = \pm 2$ with $m = \frac{n}{2}, \frac{3n}{2}, 2n$.
  In this case, for $p = 2$, equation (2.14) has the following form:

$$(3.a)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -\frac{D}{2}\varphi^4 + \frac{B}{3}\varphi^3, \qquad m = \frac{n}{2} \tag{2.24}$$

$$(3.b)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -\frac{D}{2}\varphi^4 + \frac{B}{5}\varphi, \qquad m = \frac{3n}{2} \tag{2.25}$$

$$(3.c)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -\frac{D}{2}\varphi^4 + \frac{B}{6}, \qquad m = 2n. \tag{2.26}$$

- *Case 4.* If $A = 0$, we have $p = \pm 1$ with $m = 2n, 3n$.
  Here, also we will take only the case $p = 1$; then, we will have the equations

$$(4.a)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -2D\varphi + \frac{2}{3}B\varphi^3 + C\varphi^4, \qquad m = 2n \tag{2.27}$$

$$(4.b)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = -2D\varphi + \frac{B}{2}\varphi^4 + C, \qquad m = 3n. \tag{2.28}$$

- *Case 5*. If $A = D = 0$, we have $p = \pm\frac{1}{2}$ with $m = 3n, 5n$.
  Now, for $p = \frac{1}{2}$, equation (2.14) becomes

$$(5.a)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = 2B\varphi^3 + 4C\varphi, \qquad m = 3n \tag{2.29}$$

$$(5.b)\quad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = \frac{4}{3}B\varphi^4 + 4C\varphi, \qquad m = 5n. \tag{2.30}$$

3. Traveling wave solutions for the BBM-like equations

In this section, we will obtain the solutions of the differential equations (2.17)–(2.21) in terms of the Weierstrass function, $\wp(\theta; g_2, g_3)$, in detail which allow us to get the traveling wave solutions of the $B(m, n)$ equations (1.1). Equations (2.22)–(2.30) of cases 3 and 5 can be solved in a similar way, so they will not be worked out here. Then, we will examine cases 2 and 4 since they lead to solutions in terms of the degenerate forms of the Weierstrass function. First, we will give some properties of the $\wp$ function which will be useful later [28, 29].

3.1. Relevant properties of the $\wp$ function

Let us consider a differential equation with a quartic polynomial:

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\right)^2 = P(\varphi) = a_0\varphi^4 + 4a_1\varphi^3 + 6a_2\varphi^2 + 4a_3\varphi + a_4. \tag{3.1}$$

The solution of this equation can be written in terms of the Weierstrass function where the invariants g_2 and g_3 of (3.1) are

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4 \tag{3.2}$$

and the discriminant is given by $\Delta = g_2^3 - 27g_3^2$. Then, the solution φ can be found as

$$\varphi(\theta) = \varphi_0 + \tfrac{1}{4}P_\varphi(\varphi_0)\left(\wp(\theta; g_2, g_3) - \tfrac{1}{24}P_{\varphi\varphi}(\varphi_0)\right)^{-1}, \tag{3.3}$$

where the subindex in $P_\varphi(\varphi_0)$ denotes the derivative with respect to φ, and φ_0 is one of the roots of the polynomial $P(\varphi)$ (3.1). We will have a solution with a different behavior depending on the selected root φ_0 [16].

Here, also we want to recall some other properties of the Weierstrass functions [30].

(i) The case $g_2 = 1$ and $g_3 = 0$ is called the lemniscatic case:

$$\wp(\theta; g_2, 0) = g_2^{1/2}\wp\left(\theta g_2^{1/4}; 1, 0\right), \qquad g_2 > 0. \tag{3.4}$$

The discriminant $\Delta > 0$, and the half periods have the same absolute value, where ω_1 is real and ω_2 is pure imaginary.

(ii) The case $g_2 = -1$ and $g_3 = 0$ is called the pseudo-lemniscatic case:

$$\wp(\theta; g_2, 0) = |g_2|^{1/2}\wp(\theta|g_2|^{1/4}; -1, 0), \qquad g_2 < 0. \tag{3.5}$$

For this case $\Delta < 0$ and the half periods have the same modulus with $\omega_1 = (\omega_2)^*$.

(iii) The case $g_2 = 0$ and $g_3 = 1$ is called the equianharmonic case:

$$\wp(\theta; 0, g_3) = g_3^{1/3}\wp\left(\theta g_3^{1/6}; 0, 1\right), \qquad g_3 > 0, \tag{3.6}$$

with $\Delta < 0$ and the half period ω_1 is real.

Once obtained the solution $W(\theta)$, taking into account (2.5), (2.8) and $W = \varphi^p$, the traveling wave solution of equation (1.1) is

$$u(x,t) = \phi(\xi) = W^{\frac{1}{n}}(\theta) = \varphi^{\frac{p}{n}}(\theta), \qquad \theta = \xi = hx + wt. \tag{3.7}$$

Now, we will study with certain detail the solutions of case 1 and we will give some comments about cases 2 and 4. The other cases can be worked out in a similar way.

3.2. The case $C = D = 0$, $p = -\frac{2n}{1-n}$

(1) $m = \frac{n+1}{2}$

Equation (2.17) can be written in the following form:

$$\left(\frac{d\varphi}{d\theta}\right)^2 = P(\varphi) = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{n(3n+1)}\varphi \tag{3.8}$$

and from $P(\varphi) = 0$, we get the root

$$\varphi_0 = -\frac{A(3n+1)}{2B(n+1)}. \tag{3.9}$$

The invariants (3.2) are $g_2 = g_3 = 0$ and $\Delta = 0$. Therefore, keeping in mind that $\wp(\theta; 0, 0) = \frac{1}{\theta^2}$, we find the solution of (2.17) from (3.3), where φ_0 is given by (3.9):

$$\varphi(\theta) = \frac{B^2(n-1)^2(n+1)\theta^2 - 2An(3n+1)^2}{4Bn(n+1)(3n+1)}. \tag{3.10}$$

Now, the solution of equation (1.1) obtained from (3.7) is

$$u(x,t) = \left[\frac{B^2(n-1)^2(n+1)(hx+wt)^2 - 2An(3n+1)^2}{4Bn(n+1)(3n+1)}\right]^{\frac{2}{n-1}}. \tag{3.11}$$

(2) $m = \frac{3n-1}{2}$

In this case, our equation to solve is (2.18) and the polynomial has the form

$$P(\varphi) = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{n(5n-1)}\varphi^3, \tag{3.12}$$

with one real root $\varphi_0 = \left(\frac{-A(5n-1)}{2B(n+1)}\right)^{1/3}$. Here, the discriminant $\Delta \neq 0$, and the invariants are

$$g_2 = 0, \qquad g_3 = \frac{-AB^2(n-1)^6}{32n^3(n+1)(5n-1)^2}. \tag{3.13}$$

Then, the solution of (2.18) is obtained from (3.3) for the above value of φ_0:

$$\varphi(\theta) = \varphi_0\left[\frac{4n(5n-1)\wp(\theta; 0, g_3) + 2B(n-1)^2\varphi_0}{4n(5n-1)\wp(\theta; 0, g_3) - B(n-1)^2\varphi_0}\right]. \tag{3.14}$$

By replacing (3.14) in (3.7), we get the following solution of equation (1.1):

$$u(x,t) = \left[\varphi_0^2\left(\frac{4n(5n-1)\wp(hx+wt; 0, g_3) + 2B(n-1)^2\varphi_0}{4n(5n-1)\wp(hx+wt; 0, g_3) - B(n-1)^2\varphi_0}\right)^2\right]^{\frac{1}{n-1}}, \tag{3.15}$$

with the conditions $A < 0$, $g_3 > 0$, for $\varphi_0 = \left(\frac{-A(5n-1)}{2B(n+1)}\right)^{1/3}$. For this condition on A the discriminant is negative, $\Delta < 0$, and taking into account relation (3.6), this solution can be rewritten in terms of the equianharmonic Weierstrass function:

$$u(x,t) = \left[\left(\frac{-A(5n-1)}{2B(n+1)}\right)^{2/3}\left(\frac{2^{2/3}\wp((hx+wt)g_3^{1/6}; 0, 1) + 2}{2^{2/3}\wp((hx+wt)g_3^{1/6}; 0, 1) - 1}\right)^2\right]^{\frac{1}{n-1}}. \tag{3.16}$$

(3) $m = 2n - 1$

In equation (2.19), the quartic polynomial is

$$P(\varphi) = \frac{A(n-1)^2}{2n(n+1)} + \frac{B(n-1)^2}{2n(3n-1)}\varphi^4, \tag{3.17}$$

whose two real roots are $\varphi_0 = \pm\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/4}$ for $A < 0, B > 0$ or $A > 0, B < 0$. In this case, the invariants are

$$g_2 = \frac{AB(n-1)^4}{4n^2(n+1)(3n-1)}, \qquad g_3 = 0 \tag{3.18}$$

and $\Delta < 0$. We obtain the solution of (2.19) from (3.3):

$$\varphi = \varphi_0 \left[\frac{4n(n+1)\varphi_0^2\wp(\theta; g_2, 0) - A(n-1)^2}{4n(n+1)\varphi_0^2\wp(\theta; g_2, 0) + A(n-1)^2}\right]. \tag{3.19}$$

Then, we get the solution of equation (1.1) from (3.7) with the above conditions for the real constants A, B. Therefore, keeping in mind relation (3.5), this solution can be expressed in terms of the pseudo-lemniscatic Weierstrass function:

$$u(x,t) = \left[\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}\left(\frac{2\wp((hx+wt)|g_2|^{1/4}; -1, 0) + 1}{2\wp((hx+wt)|g_2|^{1/4}; -1, 0) - 1}\right)^2\right]^{\frac{1}{n-1}} \tag{3.20}$$

for $A < 0, B > 0, g_2 < 0$ and

$$u(x,t) = \left[\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}\left(\frac{2\wp((hx+wt)|g_2|^{1/4}; -1, 0) - 1}{2\wp((hx+wt)|g_2|^{1/4}; -1, 0) + 1}\right)^2\right]^{\frac{1}{n-1}} \tag{3.21}$$

for $A > 0, B < 0, g_2 < 0$.

3.3. *The case $C = D = 0$, $p = -\frac{n}{1-n}$*

(1) $m = 2n - 1$

Now, the polynomial is cubic

$$P(\varphi) = \frac{2A(n-1)^2}{n(n+1)}\varphi + \frac{2B(n-1)^2}{n(3n-1)}\varphi^3, \tag{3.22}$$

with three distinct real roots: $\varphi_0 = 0$ and $\varphi_0 = \pm\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}$ for $A < 0, B > 0$ or $A > 0, B < 0$. Here, $\Delta > 0$ for these conditions on A, B and the invariants are

$$g_2 = \frac{-AB(n-1)^4}{n^2(n+1)(3n-1)}, \qquad g_3 = 0. \tag{3.23}$$

The solution of (2.20), obtained from (3.3), is

$$\varphi = \varphi_0 \left[\frac{2n(n+1)\varphi_0\wp(\theta; g_2, 0) - A(n-1)^2}{2n(n+1)\varphi_0\wp(\theta; g_2, 0) + A(n-1)^2}\right] \tag{3.24}$$

then, substituting (3.24) in (3.7), we get the solution of equation (1.1) with $g_2 > 0$ for $\varphi_0 = \left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}$. The other root $\varphi_0 = -\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}$ gives rise to imaginary solutions. The solution of equation (1.1) can be expressed in terms of the lemniscatic Weierstrass function using relation (3.4):

$$u(x,t) = \left[\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}\left(\frac{2\wp((hx+wt)g_2^{1/4}; 1, 0) + 1}{2\wp((hx+wt)g_2^{1/4}; 1, 0) - 1}\right)\right]^{\frac{1}{n-1}} \tag{3.25}$$

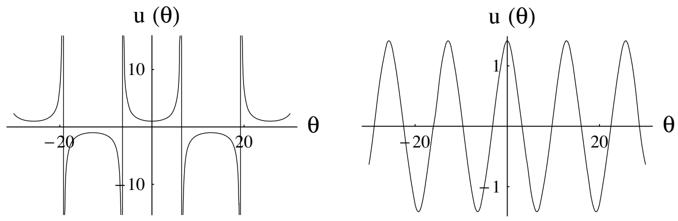

Figure 1. Case (1.c) $n = 3, m = 5$. The left figure corresponds to solution (3.20) for $h = -2$, $w = 1, a = -1$, and the right one corresponds to solution (3.21) for $h = 1, w = 1, a = -1$.

for $A < 0, B > 0, g_2 > 0$ and

$$u(x,t) = \left[\left(\frac{-A(3n-1)}{B(n+1)}\right)^{1/2}\left(\frac{2\wp((hx+wt)g_2^{1/4};1,0)-1}{2\wp((hx+wt)g_2^{1/4};1,0)+1}\right)\right]^{\frac{1}{n-1}} \tag{3.26}$$

for $A > 0, B < 0, g_2 > 0$.

(2) $m = 3n - 2$

Here, we have a quartic polynomial

$$P(\varphi) = \frac{2A(n-1)^2}{n(n+1)}\varphi + \frac{B(n-1)^2}{n(2n-1)}\varphi^4 \tag{3.27}$$

with two real roots $\varphi_0 = 0$ and $\varphi_0 = \left(-\frac{2A(2n-1)}{B(n+1)}\right)^{1/3}$. For equation (2.21), the invariants are

$$g_2 = 0, \qquad g_3 = \frac{-A^2B(n-1)^6}{4n^3(n+1)^2(2n-1)} \tag{3.28}$$

and $\Delta \neq 0$. Now, the solution of (2.21) reads from (3.3) for $\varphi_0 \neq 0$:

$$\varphi = \varphi_0\left[\frac{2n(n+1)\varphi_0\wp(\theta;0,g_3) - A(n-1)^2}{2n(n+1)\varphi_0\wp(\theta;0,g_3) + 2A(n-1)^2}\right]. \tag{3.29}$$

Then, the solution of equation (1.1) is obtained from (3.7) with the conditions: $B < 0, g_3 > 0$. Now, $\Delta < 0$ and taking into account the relation (3.6), this solution can also be expressed in terms of the equianharmonic Weierstrass function:

$$u(x,t) = \left[\left(-\frac{2A(2n-1)}{B(n+1)}\right)^{1/3}\left(\frac{2^{2/3}\wp((hx+wt)g_3^{1/6};0,1)-1}{2^{2/3}\wp((hx+wt)g_3^{1/6};0,1)+2}\right)\right]^{\frac{1}{n-1}}. \tag{3.30}$$

We have plotted these solutions except the parabolic case (3.10) for some special values of the parameters in figures 1–5. We see that the traveling waves of cases (1.a)–(1.e) have compact expressions in terms of the (pseudo) lemniscatic or equianarmonic types of the $\wp$ function. For each case the parameters (a, h, w) affect only the amplitude and scale of the wave function and therefore have a similar shape. The solutions are periodic singular for $A < 0, B > 0$ or regular for $A > 0, B < 0$.

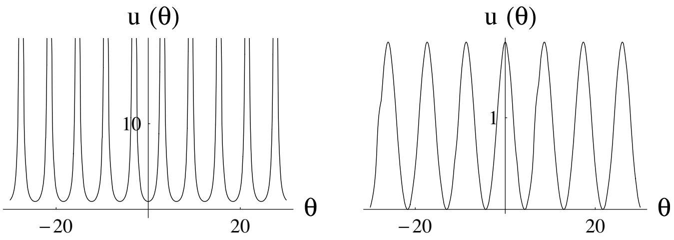

Figure 2. Case (1.c) $n = 2, m = 3$. The left figure corresponds to solution (3.20) for $h = -2$, $w = 1, a = -1$ and the right one corresponds to solution (3.21) for $h = 1, w = 1, a = -1$.

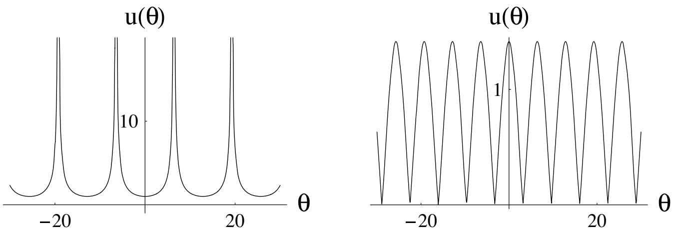

Figure 3. Case (1.d) $n = 3, m = 5$. The left figure corresponds to solution (3.25) for $h = -2$, $w = 1, a = -1$ and the right one corresponds to solution (3.26) for $h = 1, w = 1, a = -1$.

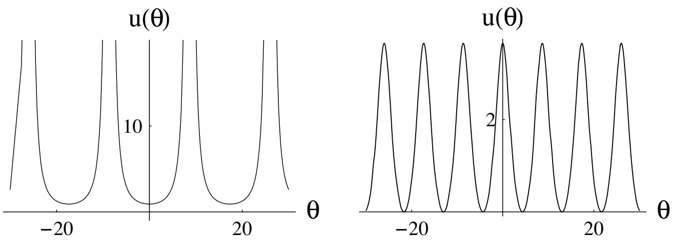

Figure 4. Case (1.d) $n = 2, m = 3$. The left figure corresponds to solution (3.25) for $h = -2$, $w = 1, a = -1$ and the right one corresponds to solution (3.26) for $h = 1, w = 1, a = -1$.

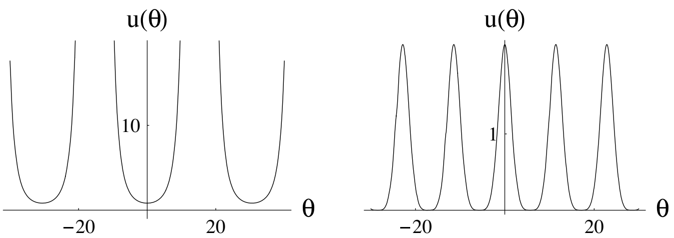

Figure 5. The left figure corresponds to solution (3.16) of case (1.b) for $h = -2, w = 1$, $a = -1, n = 2, m = 5/2$. The right one corresponds to solution (3.30) of case (1.e) for $h = 1, w = 1, a = -1, n = 3/2, m = 5/2$.

3.4. The cases $C = 0$, $p = 2$ *and* $A = 0$, $p = 1$

In this section we will give some comments only on cases (2.a), (2.b), (4.a), (4.b) being different from the others, since the discriminant can be done zero for special choices of the parameters. If $\Delta = 0$, the $\wp$ function degenerates into trigonometric or hyperbolic functions. Thus, for

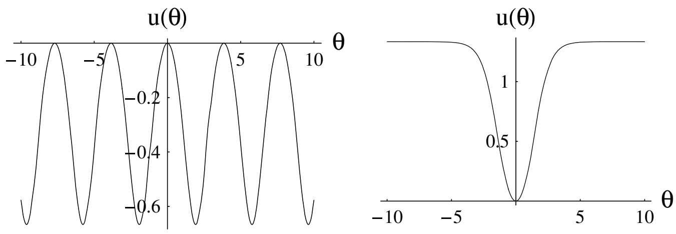

Figure 6. The left figure corresponds to the trigonometric solution of case (4.a) for $h = -1$, $w = 1$, $a = -3$, $n = 1$, $m = 2$ and the right one corresponds to the dark-soliton solution of case (4.a) for $h = -1$, $w = 1$, $a = 3$, $n = 1$, $m = 2$.

these cases we have trigonometric, hyperbolic and constant solutions. For example, in case (2.a) when we choose $B = -80A^3/729D^2$, the discriminant will be zero. Then, we have dark-soliton (hyperbolic type) and constant solutions for $A > 0$, $D < 0$ or trigonometric and constant solutions for $A < 0$, $D > 0$. For case (2.b), $\Delta = 0$, when we choose $B = -A^4/16D^3$ and we have only a constant solution. We obtain also a constant solution for (4.b) under the condition $B = 27D^4/8C^3$. In case (4.a), the condition for $\Delta = 0$ is $D = 16B^3/729C^2$. Then, we have the trigonometric and constant solutions for $A = 0$, $B > 0$ or dark-soliton, constant and hyperbolic singular solutions for $A = 0$, $B < 0$. In figure 6, we have plotted the solutions of case (4.a). It can be seen that the sign of the constant a in equation (1.1) has a drastic influence on the solutions, that is this sign changes the nature of the motion. When $a > 0$ we have hyperbolic type and when $a < 0$ we have trigonometric solutions.

4. Lagrangian and Hamiltonian

Since equation (2.9) is a motion-type, we can write the corresponding Lagrangian

$$L_W = \frac{1}{2}W_\theta^2 + \frac{An}{n+1}W^{\frac{n+1}{n}} + \frac{Bn}{m+n}W^{\frac{m+n}{n}} - DW. \tag{4.1}$$

Then, the Hamiltonian $H_W = W_\theta P_W - L_W$ reads

$$H_W(W, P_W, \theta) = \frac{1}{2}\left[P_W^2 - \left(\frac{2An}{n+1}W^{\frac{n+1}{n}} + \frac{2Bn}{m+n}W^{\frac{m+n}{n}} - 2DW\right)\right], \tag{4.2}$$

where the canonical momentum is

$$P_W = \frac{\partial L_W}{\partial W_\theta} = W_\theta. \tag{4.3}$$

The independent variable θ does not appear explicitly in (4.2); then H_W is a constant of motion, $H_W = E$,

$$E = \frac{1}{2}\left[\left(\frac{\mathrm{d}W}{\mathrm{d}\theta}\right)^2 - \left(\frac{2An}{n+1}W^{\frac{n+1}{n}} + \frac{2Bn}{m+n}W^{\frac{m+n}{n}} - 2DW\right)\right]. \tag{4.4}$$

Note that this integral of motion also leads to the first-order ODE (2.13) with the identification $C = 2E$. Now, the energy E can be expressed as a product of two independent constant of motions

$$E = \tfrac{1}{2}I_+I_-, \tag{4.5}$$

where

$$I_{\pm}(z) = \left(W_\theta \mp \sqrt{\frac{2An}{n+1} W^{\frac{n+1}{n}} + \frac{2Bn}{m+n} W^{\frac{m+n}{n}} - 2DW} \right) e^{\pm S(\theta)} \tag{4.6}$$

and the phase $S(\theta)$ is chosen in such a way that $I_{\pm}(\theta)$ be constants of motion $(\mathrm{d}I_{\pm}(\theta)/\mathrm{d}\theta = 0)$

$$S(\theta) = \int \frac{AW^{\frac{1}{n}} + BW^{\frac{m}{n}} - D}{\sqrt{\frac{2An}{n+1} W^{\frac{n+1}{n}} + \frac{2Bn}{m+n} W^{\frac{m+n}{n}} - 2DW}} \mathrm{d}\theta. \tag{4.7}$$

5. Conclusions

In this paper, we have applied the factorization technique to the $B(m, n)$ equations in order to get the traveling wave solutions. To do this, first we have performed the traveling wave reduction and then we have got the nonlinear second-order ODE that has been factorized in two ways. Then, we have considered some representative cases of the $B(m, n)$ equation for $m \neq n$. By using this method, we have obtained the traveling wave solutions in a very compact form, in terms of some special forms of the Weierstrass elliptic function: lemniscatic, pseudo-lemniscatic and equiaharmonic, where the constants appear mainly as modulating the amplitude and scale. We have also mentioned some cases giving constant, dark-soliton and trigonometric solutions. Furthermore, these solutions are not only valid for integers m and n but also for non-integer values. The case $m = n$ for the $B(m, n)$ equations has been examined by means of the factorization technique in a previous paper where the compacton and kink-like solutions were constructed [16], recovering all the solutions previously reported. Here, for $m \neq n$, solutions with compact support can also be obtained following a similar procedure. For example, in case (4.a) we have trigonometric, constant and also trivial $(u = 0)$ solutions. Two types of these solutions (trigonometric and constant solutions or trigonometric and trivial solutions) can be combined in order to get compactons. When we combine these three solutions, we can obtain the kink-like solutions. However, for cases (1.a)–(1.e) besides the non-constant solutions here obtained we can dispose only of the trivial solution $(u = 0)$, in order to construct compactons. We have also built the Lagrangian and the Hamiltonian for the second-order nonlinear ODE corresponding to the traveling wave reduction of the $B(m, n)$ equation. We have expressed the energy (4.4) as a product of two independent constant of motions. Then, we have seen that these factors are related with first-order ODEs of the factorization method that allow us to get the solutions of the nonlinear second-order ODE. In fact, whenever such autonomous systems arise, the factorization procedure is simplified due to the constant of motion. Remark that the Lagrangian underlying the nonlinear system also permits us to get solutions. There are some interesting papers showing the way to obtain compactons or kink-like traveling wave solutions of some nonlinear equations starting from the Lagrangian [31–35].

Acknowledgments

This work is partially supported by Junta de Castilla y León (Spain) Project GR224, Ministerio de Educación y Ciencia (Spain) Project FIS2009-09002 and Acción Complementaria FPA2008-04772-E/ARGEN. The author acknowledges Dr Javier Negro for useful discussions.

References

[1] Benjamin T B, Bona J L and Mahony J J 1972 *Phil. Trans. R. Soc.* A **272** 47

[2] Rosenau P and Hyman J M 1993 *Phys. Rev. Lett.* **70** 564
[3] Rosenau P 2000 *Phys. Lett.* A **275** 193
[4] Wazwaz A-M and Taha T 2003 *Math. Comput. Simul.* **62** 171
[5] Wazwaz A-M 2002 *Appl. Math. Comput.* **133** 229
[6] Wazwaz A-M 2003 *Math. Comput. Simul.* **63** 35
[7] Wazwaz A-M 2003 *Appl. Math. Comput.* **139** 37
[8] Wazwaz A-M 2006 *Chaos Soliton Fractals* **28** 454
[9] Ismail M S and Taha T R 1998 *Math. Comput. Simul.* **47** 519
[10] Ludu A and Draayer J P 1998 *Physica* D **123** 82
[11] Wazwaz A-M and Helal M A 2005 *Chaos Soliton Fractals* **26** 767
[12] Yadong S 2005 *Chaos Soliton Fractals* **25** 1083
[13] Wang L, Zhou J and Ren L 2006 *Int. J. Nonlinear Sci.* **1** 58
[14] Bruzon M S, Gandarias M L and Camacho J C 2008 *J. Nonlinear Math. Phys.* **15** 81
[15] Clarkson P A 1989 *Phys. A: Math. Gen.* **22** 3821
[16] Kuru Ş 2009 *Chaos Soliton Fractals* **42** 626
[17] Estévez P G, Kuru Ş, Negro J and Nieto L M 2009 *Chaos Soliton Fractals* **40** 2031
[18] Estévez P G, Kuru Ş, Negro J and Nieto L M 2006 *J. Phys. A: Math. Gen.* **39** 11441
[19] Cornejo-Pérez O, Negro J, Nieto L M and Rosu H C 2006 *Found. Phys.* **36** 1587
[20] Estévez P G, Kuru Ş, Negro J and Nieto L M 2007 *J. Phys. A: Math. Theor.* **40** 9819
[21] Cornejo-Pérez O 2009 *J. Phys. A: Math. Theor.* **42** 035204
[22] Wazwaz A-M 2006 *Phys. Lett.* A **355** 358
[23] Hong B and Lu D 2008 *World J. Model. and Simul.* **4** 243
[24] J-y An and W-g Zhang 2006 *Acta Math. Appl. Sin.* **22** 509
[25] Wang D-S and Li H 2008 *J. Math. Anal. Appl.* **243** 273
[26] Liu C 2006 arXiv:nlin/0609058
[27] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[28] Erdelyi A *et al* 1981 *The Bateman Manuscript Project. Higher Transcendental Functions* (Malabar: FL: Krieger)
[29] Whittaker E T and Watson G 1988 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[30] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)
[31] Arodz H 2002 *Acta Phys. Pol.* B **33** 1241
[32] Adam C, Sánchez-Guillén J and Wereszczyński A 2007 *J. Phys. A: Math. Theor.* **40** 13625
[33] Destrade M, Gaeta M and Saccomandi G 2007 *Phys. Rev.* E **75** 047601
[34] Gaeta G, Gramchev T and Walcher S 2007 *J. Phys. A: Math. Theor.* **40** 4493
[35] Gaeta G 2007 *Europhys. Lett.* **79** 20003